

Random ball-polyhedra and inequalities for intrinsic volumes

Grigoris Paouris*

Peter Pivovarov[†]

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Abstract

We prove a randomized version of the generalized Urysohn inequality relating mean width to the other intrinsic volumes. To do this, we introduce a stochastic approximation procedure that sees each convex body K as the limit of intersections of Euclidean balls of large radii and centered at randomly chosen points. The proof depends on a new isoperimetric inequality for the intrinsic volumes of such intersections. If the centers are i.i.d. and sampled according to a bounded continuous distribution, then the extremizing measure is uniform on a Euclidean ball. If one additionally assumes that the centers have i.i.d. coordinates, then the uniform measure on a cube is the extremizer. We also discuss connections to a randomized version of the extended isoperimetric inequality and symmetrization techniques.

1 Introduction

In this paper we discuss stochastic forms of classical inequalities for intrinsic volumes. Recall that the intrinsic volumes V_1, \dots, V_n are functionals on convex bodies which can be defined via the Steiner formula: for any convex body $K \subseteq \mathbb{R}^n$ and $\varepsilon > 0$,

$$|K + \varepsilon B| = \sum_{j=0}^n \omega_{n-j} V_j(K) \varepsilon^{n-j},$$

where $|\cdot|$ denotes n -dimensional Lebesgue measure, $B = B_2^n$ is the unit Euclidean ball in \mathbb{R}^n , ω_{n-j} is the volume of B_2^{n-j} , and $V_0 \equiv 1$; V_1 is a multiple of the mean

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width w , $2V_{n-1}$ is the surface area and $V_n = |\cdot|$ is the volume. The V_j 's satisfy the extended isoperimetric inequality: for $1 \leq j < n$,

$$\left(\frac{V_n(K)}{V_n(B)} \right)^{1/n} \leq \left(\frac{V_j(K)}{V_j(B)} \right)^{1/j}; \quad (1.1)$$

as well as the generalized Urysohn inequality: for $1 < j \leq n$,

$$\left(\frac{V_j(K)}{V_j(B)} \right)^{1/j} \leq \frac{V_1(K)}{V_1(B)}. \quad (1.2)$$

The classical isoperimetric inequality corresponds to $j = n-1$ in (1.1); Urysohn's inequality to $j = n$ in (1.2) (or $j = 1$ in (1.1)). The Alexandrov-Fenchel inequality for mixed volumes (e.g. [36]) implies both (1.1) and (1.2). Alternatively, symmetrization methods can be used. For example, Steiner symmetrization, which preserves $V_n(K)$ and does not increase $V_j(K)$, can be used to prove (1.1); a general framework for such inequalities, building on work of Rogers and Shephard [34], is discussed by Campi and Gronchi in [12]. On the other hand, Minkowski symmetrization, which fixes $V_1(K)$ and does not increase $V_j(K)$, can be used to prove (1.2); see [21, §6.4.4] (§2 contains definitions of these symmetrizations).

Both Steiner and Minkowski symmetrization can also be applied in suitable stochastic settings and yield stronger forms of such inequalities for associated random convex sets. For example, known inequalities for the expected intrinsic volumes of random convex hulls lead to (1.1). Such inequalities have their roots in the classical Sylvester's four point problem, e.g., [31], and build on work of Busemann [11], Groemer [19], Rogers-Shephard [34], Pfiefer [30], Campi-Gronchi [12], Hartzoulaki-Paouris [22], among others. Drawing on [28], one can formulate a type of stochastic dominance as follows. Assume that $|K| = |B|$ and sample independent random vectors X_1, \dots, X_N according to the uniform density $\frac{1}{|K|}\mathbb{1}_K$, i.e., $\mathbb{P}(X_i \in A) = \frac{1}{|K|} \int_A \mathbb{1}_K(x) dx$ for Borel sets $A \subseteq \mathbb{R}^n$. Additionally, sample independent random vectors Z_1, \dots, Z_N according to $\frac{1}{|B|}\mathbb{1}_B$. Then for all $1 \leq j \leq n$ and $s > 0$,

$$\mathbb{P}\left(V_j(\text{conv}\{X_1, \dots, X_N\}) > s\right) \geq \mathbb{P}\left(V_j(\text{conv}\{Z_1, \dots, Z_N\}) > s\right), \quad (1.3)$$

where conv denotes the convex hull. Integrating in s yields

$$\mathbb{E}V_j(\text{conv}\{X_1, \dots, X_N\}) \geq \mathbb{E}V_j(\text{conv}\{Z_1, \dots, Z_N\}). \quad (1.4)$$

By the law of large numbers, the latter convex hulls converge to their respective ambient bodies and thus when $N \rightarrow \infty$, $V_j(K) \geq V_j(B)$ whenever $V_n(K) = V_n(B)$,

which is equivalent to (1.1). Thus (1.1) can be seen as a global inequality which arises through a random approximation procedure in which stochastic domination holds at each stage. In fact, (1.3) holds not just for the convex hull but for a variety of other (linear, convex) operations and one can sample points according to continuous distributions on \mathbb{R}^n (see [28]). Such distributional inequalities are useful for small deviation inequalities for the volume of random sets [29]; inequalities in the dual setting, obtained in joint work with Cordero-Erausquin and Fradelizi [14], lead to a stochastic version of the Blaschke-Santaló inequality and the L_p -versions of Lutwak and Zhang [26]. All of these inequalities concern volume and can be proved by Steiner symmetrization in an appropriate setting.

In a natural dual setting, Böröczky and Schneider [9] studied intersections of random halfspaces according to the following model. Let \mathcal{H} denote the collection of all affine hyperplanes in \mathbb{R}^n with its usual topology. Given a convex body K with $V_1(K) = V_1(B)$, let \mathcal{H}_K be the collection of $H \in \mathcal{H}$ that meet $K + B$ but do not intersect the interior of K . Let μ be the canonical motion invariant Borel measure on \mathcal{H} normalized so that $\mu(\{H \in \mathcal{H} : H \cap M \neq \emptyset\})$ is the mean width $w(M)$ of M , for each convex body $M \subseteq \mathbb{R}^n$. Let $2\mu_K$ be the restriction of μ to \mathcal{H}_K so that μ_K is a probability measure. Sample independent hyperplanes H_1, \dots, H_N according to μ_K and J_1, \dots, J_N according to μ_B . Denoting by H_i^- the closed halfspace bounded by H_i and containing K , same for J_i^- and B , the following inequality holds for expectations

$$\mathbb{E}V_j\left(\bigcap_{i=1}^N H_i^- \cap (K+B)\right)^{1/j} \leq \mathbb{E}V_j\left(\bigcap_{i=1}^N J_i^- \cap (2B)\right)^{1/j} \quad (1.5)$$

(their result is stated only for $j = 1$ but the proof applies to all $1 \leq j \leq n$; the proof is reproduced in §5). When $N \rightarrow \infty$, one obtains $V_j(K) \leq V_j(B)$ whenever $V_1(K) = V_1(B)$, which is equivalent to (1.2). The proof of (1.5) uses Minkowski symmetrization. We are not aware of extensions of (1.5) to higher moments or for stochastic dominance.

In this paper we study a different model of random sets and a connection to (1.2) for which there is an underlying stochastic dominance. In [7], Bezdek, Lángi, Naszódi, and Papez study the intersection of finitely many (unit) Euclidean balls, called *ball-polyhedra*, and lay out a broad framework for their study; they treat analogues of classical theorems in convexity such as those of Carathéodory and Steinitz, and they study their facial structure. Motivation for their study arises, in part, from the Kneser-Poulsen Conjecture on the monotonicity of the volume of intersections (or unions) of Euclidean balls under contractions of their centers; see e.g. Bezdek's expository monograph [6]. We

consider intersections of balls of a given radius $R > 0$ with centers X_i that are sampled independently according to a continuous distribution, i.e., a density $f : \mathbb{R}^n \rightarrow [0, \infty)$ with $\int_{\mathbb{R}^n} f(x) dx = 1$ so that $\mathbb{P}(X_i \in A) = \int_A f(x) dx$ for Borel sets $A \subseteq \mathbb{R}^n$. In what follows, by a *probability density* we always mean that of a continuous distribution. Different random models associated with ball-polyhedra have been studied by Csikós [15], Ambrus, Kevei and Vígh [1] and Fodor, Kevei and Vígh [17].

Our first result is the following isoperimetric inequality for intrinsic volumes; here $B(x, r)$ is the closed Euclidean ball in \mathbb{R}^n centered at $x \in \mathbb{R}^n$ with radius $r > 0$ (so $B = B(0, 1)$).

Theorem 1.1. *Let $N, n \geq 1$ and $R > 0$. Let f be a probability density on \mathbb{R}^n that is bounded by one. Consider independent random vectors X_1, \dots, X_N sampled according to f and Z_1, \dots, Z_N according to $\mathbb{1}_{B(0, r_n)}$ where $r_n > 0$ is chosen so that $|B(0, r_n)| = 1$. Then for all $1 \leq j \leq n$ and $s > 0$,*

$$\mathbb{P}\left(V_j\left(\bigcap_{i=1}^N B(X_i, R)\right) > s\right) \leq \mathbb{P}\left(V_j\left(\bigcap_{i=1}^N B(Z_i, R)\right) > s\right). \quad (1.6)$$

For a particular choice of density f , (1.6) can be seen as a form of (1.2) in which stochastic dominance holds. The connection arises from the following: given a convex body $K \subseteq \mathbb{R}^n$ with the origin in its interior and $K \subseteq B(0, R)$, define a star-shaped set $A(K, R)$ with radial function $\rho_{A(K, R)}(-\theta) = R - h_K(\theta)$ (see §2 for definitions). Euclidean balls centered at points in $A(K, R)$ are tangent to hyperplanes that meet $B(0, R)$ but not the interior of K . By choosing $f = \frac{1}{|A(K, R)|} \mathbb{1}_{A(K, R)}$ in Theorem (1.1), we get the following corollary.

Corollary 1.2. *Let K be a convex body in \mathbb{R}^n with the origin in its interior, $R > 0$ and assume $K \subseteq B(0, R)$. Consider independent random vectors X_1, \dots, X_N sampled according to $\frac{1}{|A(K, R)|} \mathbb{1}_{A(K, R)}$ and Z_1, \dots, Z_N according to $\frac{1}{|rB|} \mathbb{1}_{rB}$, where $r = r(K, n, R)$ satisfies $|A(K, R)| = |rB|$. Then for each $p \in \mathbb{R}$,*

$$\left(\mathbb{E} V_j\left(\bigcap_{i=1}^N B(X_i, R)\right)^p\right)^{1/p} \leq \left(\mathbb{E} V_j\left(\bigcap_{i=1}^N B(Z_i, R)\right)^p\right)^{1/p}. \quad (1.7)$$

For large R the intersection of such balls resembles intersections of halfspaces; it turns out that the volume normalization $|A(K, R)| = |rB|$ amounts to a constraint on the mean width of K . When $N \rightarrow \infty$ and $R \rightarrow \infty$ in (1.7), we get (1.2). In fact, for fixed N , when $p \rightarrow -\infty$ and $R \rightarrow \infty$, we obtain the following result: among all convex bodies K of a given mean width, the minimal j -th intrinsic volume of the intersection of $N > n$ halfspaces containing K is maximized when K is a ball; the latter is a special case of a result of Schneider [35], which is also proved using Minkowski symmetrization.

The proof of Theorem 1.1 draws on both symmetrization techniques discussed above. We use the fact that $K \mapsto V_j(K)^{1/j}$ is concave with respect to Minkowski addition, which follows by Minkowski symmetrization [21, §6.4.4], or the Alexandrov-Fenchel inequalities. However, using Steiner symmetrization and rearrangement inequalities, which are typically applied to volumetric inequalities, we obtain a distributional form of (1.2), which for $j < n$ is not a volumetric inequality. We make essential use of continuous distributions and intersections of Euclidean balls, as opposed to intersections of translates of other convex bodies or halfspaces (see Remark 3.7). Another fundamental ingredient in our proof is Kanter's theorem from [24] on stochastic dominance for products of unimodal densities, which we have not used before in this context. The techniques used in the proof of Theorem 1.1 also apply when V_j is replaced by a function ϕ which is invariant under rotations, monotone and *quasi-concave* with respect to Minkowski addition (see Theorem 3.1).

As mentioned, (1.1) and (1.2) share a common result - Urysohn's inequality. We have discussed three randomized inequalities that have Urysohn's inequality as a consequence: for random convex hulls by taking $j = 1$ in (1.4); for random halfspaces by taking $j = n$ in (1.5); for random ball-polyhedra by taking $j = n$ in (1.7). It is natural to investigate the relationship between the randomized forms. We note that the random ball-polyhedra version implies the random convex hull version. This is a consequence of a result of Gorbovickis [18], used to establish the Kneser-Poulsen conjecture for large radii (see §5).

We also consider random ball-polyhedra with independently chosen centers $X_i = (X_{i1}, \dots, X_{in}) \in \mathbb{R}^n$ having independent coordinates and bounded densities, say by one. In this case, the uniform density on the unit cube $Q_n = [-1/2, 1/2]^n$ is the extremizer.

Theorem 1.3. *Let $N, n \geq 1$ and $R > 0$. Let $h(x) = \prod_{i=1}^n h_i(x_i)$, where each h_i is a probability density on \mathbb{R} that is bounded by one. Consider independent random vectors X_1, \dots, X_N sampled according to h and Y_1, \dots, Y_N according to $\mathbb{1}_{Q_n}$. Then for all $1 \leq j \leq n$ and $s > 0$,*

$$\mathbb{P}\left(V_j\left(\bigcap_{i=1}^N B(X_i, R)\right) > s\right) \leq \mathbb{P}\left(V_j\left(\bigcap_{i=1}^N B(Y_i, R)\right) > s\right). \quad (1.8)$$

Lastly, on the organization of the paper: we recall definitions in §2. Theorems 1.1 and 1.3 are proved in §3. In §4, we recall the definition of the Wulff shape and discuss a connection to (non-random) ball-polyhedra. In §5, we prove Corollary 1.2 and compare it to kindred results for intersections of halfspaces, including a numerical improvement on the minimal volume simplex containing a convex body due to Kanazawa [23]; we also discuss Minkowski

symmetrization, and compare two random versions of Urysohn's inequality.

2 Preliminaries

We work in Euclidean space \mathbb{R}^n with the canonical inner product $\langle \cdot, \cdot \rangle$, Euclidean norm $|\cdot|$; we also use $|\cdot|$ (or V_n) for volume. As above, the unit Euclidean ball in \mathbb{R}^n is $B = B_2^n$ and its volume is $\omega_n := |B_2^n|$; S^{n-1} is the unit sphere, equipped with the Haar probability measure σ .

A convex body $K \subseteq \mathbb{R}^n$ is a compact, convex set with non-empty interior. The set of all convex bodies in \mathbb{R}^n is denoted by \mathcal{K}^n . For $K, L \in \mathcal{K}^n$, the Minkowski sum $K + L$ is the set $\{x + y : x \in K, y \in L\}$; for $\alpha > 0$, $\alpha K = \{\alpha x : x \in K\}$. We say that K is symmetric if it is symmetric about the origin, i.e., $-x \in K$ whenever $x \in K$. For $K \in \mathcal{K}^n$, the support function of K is given by

$$h_K(x) = \sup\{\langle y, x \rangle : y \in K\} \quad (x \in \mathbb{R}^n).$$

The mean width of K is

$$w(K) = \int_{S^{n-1}} h_K(\theta) + h_K(-\theta) d\sigma(\theta) = 2 \int_{S^{n-1}} h_K(\theta) d\sigma(\theta).$$

If $K \in \mathcal{K}^n$ and $u \in S^{n-1}$, the Minkowski symmetral of K about u^\perp is the convex body

$$M_u(K) = \frac{K + R_u(K)}{2},$$

where R_u is the reflection about u^\perp . The Steiner symmetral of a convex body will be defined later, and more generally for functions.

For compact sets C_1, C_2 in \mathbb{R}^n , we let $\delta^H(C_1, C_2)$ denote Hausdorff distance:

$$\delta^H(C_1, C_2) = \inf\{\varepsilon > 0 : C_1 \subseteq C_2 + \varepsilon B_2^n, C_2 \subseteq C_1 + \varepsilon B_2^n\}.$$

A set $K \subseteq \mathbb{R}^n$ is star-shaped if it is compact, contains the origin in its interior and for every $x \in K$ and $\lambda \in [0, 1]$ we have $\lambda x \in K$. We call K a star-body if its radial function

$$\rho_K(\theta) = \sup\{s > 0 : s\theta \in K\} \quad (\theta \in S^{n-1})$$

is positive and continuous. Any positive continuous function $f : S^{n-1} \rightarrow \mathbb{R}$ determines a star body with radial function f .

For non-negative functions f and g on $[0, \infty)$, we write $f(r) = O(g(r))$ as $r \rightarrow \infty$ if there exists $M > 0$ and $r_0 > 0$ such that $f(r) \leq M g(r)$ for all $r \geq r_0$; we write $f(r) = o(g(r))$ if $f(r)/g(r) \rightarrow 0$ as $r \rightarrow \infty$.

We say that a non-negative function f on \mathbb{R}^n is quasi-concave if $\{x \in \mathbb{R}^n : f(x) > s\}$ is convex for each $s \geq 0$.

For Borel sets $A \subseteq \mathbb{R}^n$ with $|A| < \infty$, the volume-radius $\text{vr}(A)$ is the radius of a Euclidean ball with the same volume as A ; the symmetric rearrangement A^* of A is the (open) Euclidean ball of radius $\text{vr}(A)$ centered at the origin. The symmetric decreasing rearrangement of 1_A is defined by $(1_A)^* := 1_{A^*}$. If $f : \mathbb{R}^n \rightarrow \mathbb{R}^+$ is an integrable function, we define its symmetric decreasing rearrangement f^* by

$$f^*(x) = \int_0^\infty 1_{\{f>s\}}^*(x) ds = \int_0^\infty 1_{\{f>s\}^*}(x) ds.$$

The latter should be compared with the “layer-cake representation” of f :

$$f(x) = \int_0^\infty 1_{\{f>s\}}(x) ds; \quad (2.1)$$

see [25, Theorem 1.13]. The function f^* is radially symmetric, radially decreasing (henceforth we simply say ‘decreasing’) and equimeasurable with f , i.e., $\{f > s\}$ and $\{f^* > s\}$ have the same volume for each $s > 0$. By equimeasurability one has $\|f\|_p = \|f^*\|_p$ for each $1 \leq p \leq \infty$, where $\|\cdot\|_p$ denotes the $L_p(\mathbb{R}^n)$ -norm. For a nonnegative, integrable function f on \mathbb{R}^n , the rearrangement f^* can be reached by a sequence of *Steiner symmetrals* $f^*(\cdot|\theta)$, which correspond to symmetrization in dimension one in the direction $\theta \in S^{n-1}$; namely $f^*(\cdot|\theta)$ is obtained by rearranging f along every line parallel to θ . The function $f^*(\cdot|\theta)$ is symmetric with respect to θ^\perp . We refer the reader to the book [25] for further background material on rearrangements of functions.

3 Extremal inequalities for random ball-polyhedra

In this section we prove a more general version of Theorem 1.1. It concerns a family of functions $\phi : \mathcal{K}^n \rightarrow [0, \infty)$ satisfying the following three conditions:

- (a) *quasi-concave with respect to Minkowski addition*: for all $K, L \in \mathcal{K}^n$ and $\lambda \in (0, 1)$,

$$\phi((1 - \lambda)K + \lambda L) \geq \min(\phi(K), \phi(L));$$

- (b) *monotone*: $\phi(K) \leq \phi(L)$ whenever $K, L \in \mathcal{K}^n$ satisfy $K \subseteq L$;
- (c) *rotation-invariant*: $\phi(UK) = \phi(K)$ for all orthogonal transformations U of \mathbb{R}^n and $K \in \mathcal{K}^n$.

The concavity of $K \mapsto V_j(K)^{1/j}$ can be proved using Minkowski symmetrization as in [21, §6.4.4] or as a consequence of the Alexandrov-Fenchel inequalities; V_j also satisfies (b) and (c) (see [36] for background).

Theorem 3.1. *Let $N, n \geq 1$ and $r_1, \dots, r_N \in (0, \infty)$. Assume that $\phi : \mathcal{K}^n \rightarrow [0, \infty)$ satisfies (a), (b) and (c). Let f_1, \dots, f_N be probability densities on \mathbb{R}^n . Consider independent random vectors X_1, \dots, X_N and X_1^*, \dots, X_N^* such that X_i is distributed according to f_i and X_i^* according to f_i^* , for $i = 1, \dots, N$. Then for any $s \geq 0$,*

$$\mathbb{P}\left(\phi\left(\bigcap_{i=1}^N B(X_i, r_i)\right) > s\right) \leq \mathbb{P}\left(\phi\left(\bigcap_{i=1}^N B(X_i^*, r_i)\right) > s\right). \quad (3.1)$$

Furthermore, assume each f_i is bounded. Let Z_1, \dots, Z_N be independent random vectors with Z_i distributed according to $a_i \mathbb{1}_{b_i B}$, where $a_i = \|f_i\|_\infty$ and b_i satisfies $\int_{\mathbb{R}^n} a_i \mathbb{1}_{b_i B} dx = 1$, for $i = 1, \dots, N$. Then

$$\mathbb{P}\left(\phi\left(\bigcap_{i=1}^N B(X_i, r_i)\right) > s\right) \leq \mathbb{P}\left(\phi\left(\bigcap_{i=1}^N B(Z_i, r_i)\right) > s\right). \quad (3.2)$$

As in [28], [29], we use the rearrangement inequality of Rogers [33] and Brascamp-Lieb-Luttinger [10]; in particular, the following variant due to Christ [13].

Theorem 3.2. *Let $F : (\mathbb{R}^n)^N = \otimes_{i=1}^N \mathbb{R}^n \rightarrow [0, \infty)$. Then*

$$\begin{aligned} \int_{(\mathbb{R}^n)^N} F(x_1, \dots, x_N) f_1(x_1) \cdots f_N(x_N) dx_1 \dots dx_N \\ \leq \int_{(\mathbb{R}^n)^N} F(x_1, \dots, x_N) f_1^*(x_1) \cdots f_N^*(x_N) dx_1 \dots dx_N \end{aligned} \quad (3.3)$$

holds for any integrable $f_1, \dots, f_N : \mathbb{R}^n \rightarrow [0, \infty)$ whenever F satisfies the following condition: for every $z \in S^{n-1} \subseteq \mathbb{R}^n$ and for every $Y = (y_1, \dots, y_N) \subseteq (z^\perp)^N \subseteq (\mathbb{R}^n)^N$, the function $F_{z,Y} : \mathbb{R}^N \rightarrow [0, \infty)$ defined by

$$F_{z,Y}(t) := F(y_1 + t_1 z, \dots, y_N + t_N z). \quad (3.4)$$

is even and quasi-concave.

Remark 3.3. (i) When $n = 1$, the condition on F in the latter theorem reduces to $F : \mathbb{R}^N \rightarrow [0, \infty)$ being even and quasi-concave.

(ii) The proof of the latter theorem relies on the fact that such integrals are increased when the f_i 's are replaced by their Steiner symmetrals $f_i^*(\cdot|\theta)$.

When repeated in suitable directions θ , they yield the symmetric decreasing rearrangements f_i^* . We refer the reader to [13] or [28], [14] for the details.

We also combine the latter with a theorem of Kanter [24, Corollary 3.2]. If f and g are probability densities on \mathbb{R}^n such that $\int_K f(x)dx \leq \int_K g(x)dx$ for every symmetric convex set $K \subseteq \mathbb{R}^n$, we will use similar terminology as that used in [4], [5], [3] and say that f is less peaked than g (here, as above, ‘symmetric’ means ‘origin-symmetric’). Furthermore, we say that f is unimodal if it is quasi-concave and even. Kanter uses a more general notion of unimodality but his result applies to the condition we use here; see the discussion in [24, §5].

Theorem 3.4. *Let $n, N \geq 1$. Let f_1, \dots, f_N and g_1, \dots, g_N be unimodal probability densities on \mathbb{R}^n . Assume that f_i is less peaked than g_i for each $i = 1, \dots, N$. Then $\prod_{i=1}^n f_i$ is less peaked than $\prod_{i=1}^n g_i$.*

We will also use the following basic lemma (it can be proved using, e.g., [14, Lemma 4.3]).

Lemma 3.5. *Any radial probability density on \mathbb{R}^n , $n \geq 1$, that is bounded by one is less peaked than $\mathbb{1}_{B(0, r_n)}$ where r_n satisfies $|B(0, r_n)| = 1$; in particular, taking $n = 1$, any even probability density on \mathbb{R} that is bounded by one is less peaked than $\mathbb{1}_{[-1/2, 1/2]}$.*

The requisite quasi-concavity needed to apply Theorem 3.2 is a consequence of the following lemma.

Lemma 3.6. *Let $N, n \geq 1$ and $r_1, \dots, r_N \in (0, \infty)$. Assume that $\phi : \mathcal{K}^n \rightarrow [0, \infty)$ satisfies (a) and (b) and $\phi(K) = \phi(-K)$ for each $K \in \mathcal{K}^n$. Set*

$$F(x_1, \dots, x_N) = \phi\left(\bigcap_{i=1}^N B(x_i, r_i)\right).$$

Then F is even and quasi-concave on its support. Additionally, assume that ϕ satisfies condition (c). If $z \in S^{n-1}$ and $y_1, \dots, y_N \in z^\perp$ and $F_{z,Y} : \mathbb{R}^N \rightarrow [0, \infty)$ is defined by

$$F_{z,Y}(t) := \phi\left(\bigcap_{i=1}^N B(y_i + t_i z, r_i)\right),$$

then $F_{z,Y}$ is even and quasi-concave on its support.

Proof. The function F is clearly even on $(\mathbb{R}^n)^N$. For the quasi-concavity claim, let $\mathbf{u} = (u_1, \dots, u_N) \in (\mathbb{R}^n)^N$ and $\mathbf{v} = (v_1, \dots, v_N) \in (\mathbb{R}^n)^N$ belong to the support of F . We will first show that

$$\bigcap_{i=1}^N B\left(\frac{u_i + v_i}{2}, r_i\right) \supseteq \frac{1}{2} \bigcap_{i=1}^N B(u_i, r_i) + \frac{1}{2} \bigcap_{i=1}^N B(v_i, r_i).$$

Let $w_1, w_2 \in \mathbb{R}^n$ and assume $|w_1 - u_i| \leq r_i$ and $|w_2 - v_i| \leq r_i$ for $i = 1, \dots, N$. Then for $i = 1, \dots, N$,

$$\left| \frac{w_1 + w_2}{2} - \left(\frac{u_i + v_i}{2} \right) \right| \leq \frac{1}{2}|w_1 - u_i| + \frac{1}{2}|w_2 - v_i| \leq r_i,$$

which shows the inclusion. By monotonicity and quasi-concavity of ϕ , we have

$$\begin{aligned} F((\mathbf{u} + \mathbf{v})/2) &= \phi\left(\bigcap_{i=1}^N B\left(\frac{u_i + v_i}{2}, r_i\right)\right) \\ &\geq \phi\left(\frac{1}{2} \bigcap_{i=1}^N B(u_i, r_i) + \frac{1}{2} \bigcap_{i=1}^N B(v_i, r_i)\right) \\ &\geq \min\left(\phi\left(\bigcap_{i=1}^N B(u_i, r_i)\right), \phi\left(\bigcap_{i=1}^N B(v_i, r_i)\right)\right) \\ &= \min(F(\mathbf{u}), F(\mathbf{v})). \end{aligned}$$

Therefore, F is quasi-concave on its support.

The second quasi-concavity claim follows from the fact that the restriction of a quasi-concave function to a line is itself quasi-concave. Finally, let $z \in S^{n-1}$ and $y_1, \dots, y_N \in z^\perp$. Let R_z denote the reflection about z^\perp . Then

$$\begin{aligned} R_z\left(\bigcap_{i=1}^N B(y_i + t_i z, r_i)\right) &= \bigcap_{i=1}^N R_z(r_i B(0, 1) + (y_i + t_i z)) \\ &= \bigcap_{i=1}^N (r_i B(0, 1) + (y_i - t_i z)) \\ &= \bigcap_{i=1}^N B(y_i - t_i z, r_i). \end{aligned}$$

Since ϕ satisfies (c), we have

$$F_{z,Y}(t) = \phi\left(\bigcap_{i=1}^N B(y_i + t_i z, r_i)\right) = \phi\left(\bigcap_{i=1}^N B(y_i - t_i z, r_i)\right) = F_{z,Y}(-t).$$

□

Remark 3.7. In the previous lemma, the use of Euclidean balls is not important to obtain quasi-concavity of F ; one can also take intersections of translates of other convex bodies. However, to obtain the evenness condition on $F_{z,Y}$ it is essential that we use Euclidean balls. Thus ball-polyhedra interface well with the rearrangement techniques used here, as the next proof shows.

Proof of Theorem 3.1. Let F be as in Lemma 3.6. For $s > 0$, set $H = \mathbb{1}_{\{F > s\}}$. Let $z \in S^{n-1}$ and $Y = (y_1, \dots, y_N) \in (z^\perp)^N$. Let $F_{z,Y}(t_1, \dots, t_N) = F(y_1 + t_1 z, \dots, y_N + t_N z)$

and $H_{z,Y}(t_1, \dots, t_N) = \mathbb{1}_{\{F_{z,Y} > s\}}(t_1, \dots, t_N)$. By Lemma 3.6, $F_{z,Y}$ is an even, quasi-concave function. It follows that $H_{z,Y}$ is even and quasi-concave. Therefore we can apply Theorem 3.2 to obtain

$$\begin{aligned} & \mathbb{P}\left(\phi\left(\bigcap_{i=1}^N B(X_i, r_i)\right) > s\right) \\ &= \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} H(x_1, \dots, x_N) \prod_{i=1}^N f_i(x_i) dx_1 \dots dx_N \\ &\leq \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} H(x_1, \dots, x_N) \prod_{i=1}^N f_i^*(x_i) dx_1 \dots dx_N \\ &= \mathbb{P}\left(\phi\left(\bigcap_{i=1}^N B(X_i^*, r_i)\right) > s\right), \end{aligned}$$

which proves (3.1).

We will first prove (3.2) under the additional assumption that $\|f_i\|_\infty = 1$ for $i = 1, \dots, N$. Furthermore, by the first part of the proof we may assume that each f_i is radially symmetric and radially decreasing, hence unimodal. By Lemma 3.5, f_i is less peaked than $\mathbb{1}_{B(0, r_n)}$. Since $H = \mathbb{1}_{\{F > s\}}$ is the indicator function of a symmetric convex set in $(\mathbb{R}^n)^N$, Theorem 3.4 yields

$$\begin{aligned} & \mathbb{P}\left(\phi\left(\bigcap_{i=1}^N B(X_i, r_i)\right) > s\right) \\ &= \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} H(x_1, \dots, x_N) \prod_{i=1}^N f_i(x_i) dx_1 \dots dx_N \\ &\leq \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} H(x_1, \dots, x_N) \prod_{i=1}^N \mathbb{1}_{B(0, r_n)}(x_i) dx_1 \dots dx_N \\ &= \mathbb{P}\left(\phi\left(\bigcap_{i=1}^N B(Z_i, r_i)\right) > s\right). \end{aligned}$$

The general case follows by a change of variables; note that we make no assumption of homogeneity of ϕ in the following argument. For $i = 1, \dots, N$, let $c_i = \|f_i\|_\infty^{-1/n}$ and set

$$\bar{f}_i(x) = \frac{f_i(c_i x)}{\int_{\mathbb{R}^n} f_i(c_i y) dy} = \frac{f_i(c_i x)}{\|f_i\|_\infty}.$$

Then $\|\bar{f}_i\|_1 = \|\bar{f}_i\|_\infty = 1$ for $i = 1, \dots, N$. We apply what we just proved with $\bar{f}_1, \bar{f}_2, \dots, \bar{f}_N$ and $H(c_1 \cdot, \dots, c_N \cdot)$ (which remains the indicator of a symmetric convex set)

$$\mathbb{P}\left(\phi\left(\bigcap_{i=1}^N B(X_i, r_i)\right) > s\right)$$

$$\begin{aligned}
&= \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} H(x_1, \dots, x_N) \prod_{i=1}^N f_i(x_i) dx_1 \dots dx_N \\
&= \prod_{i=1}^N \|f_i\|_\infty \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} H(c_1 y_1, \dots, c_n y_N) \prod_{i=1}^N \frac{c_i^n f_i(c_i y_i)}{\|f_i\|_\infty} dy_1 \dots dy_N \\
&= \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} H(c_1 y_1, \dots, c_n y_N) \prod_{i=1}^N \tilde{f}_i(y_i) dy_1 \dots dy_N \\
&\leq \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} H(c_1 y_1, \dots, c_n y_N) \prod_{i=1}^N \mathbb{1}_{r_n B}(y_i) dy_1 \dots dy_N \\
&= \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} H(x_1, \dots, x_N) \prod_{i=1}^N \|f_i\|_\infty \mathbb{1}_{c_i r_n B}(x_i) dx_1 \dots dx_N \\
&= \mathbb{P}\left(\phi\left(\bigcap_{i=1}^N B(Z_i, r_i)\right) > s\right),
\end{aligned}$$

where, as above, $r_n = \omega_n^{-1/n}$. This proves (3.2) as claimed with $b_i = c_i r_n$, for $i = 1, \dots, N$. \square

Now we turn to a generalization of Theorem 1.3.

Theorem 3.8. *Let $N, n \geq 1$ and $r_1, \dots, r_N \in (0, \infty)$. Assume that $\phi : \mathcal{K}^n \rightarrow [0, \infty)$ satisfies (a) and (b). Let h_1, \dots, h_N be probability densities on \mathbb{R}^n with $h_i(x) = \prod_{j=1}^n h_{ij}(x_j)$ and each h_{ij} is a probability density on \mathbb{R} that is bounded by one. Consider independent random vectors X_1, \dots, X_N and Y_1, \dots, Y_N such that X_i is distributed according to h_i and Y_i according to $\mathbb{1}_{Q_n}$, for $i = 1, \dots, N$. Then for any $s \geq 0$,*

$$\mathbb{P}\left(\phi\left(\bigcap_{i=1}^N B(X_i, r_i)\right) > s\right) \leq \mathbb{P}\left(\phi\left(\bigcap_{i=1}^N B(Y_i, r_i)\right) > s\right). \quad (3.5)$$

Proof. Note that each h_{ij}^* is less peaked than $\mathbb{1}_{[-1/2, 1/2]}$, hence by Theorem 3.4, $\prod_{i=1}^N \prod_{j=1}^n h_{ij}^*$ is less peaked than $\prod_{i=1}^N \mathbb{1}_{Q_n}$. Let F be as in Lemma 3.6, $s > 0$ and $H = \mathbb{1}_{\{F > s\}}$. For $x_i \in \mathbb{R}^n$ we write $x_i = (x_{i1}, \dots, x_{in})$. Since F is even and quasi-concave on its support, we can apply Theorem 3.2 (considering F as a quasi-concave function on \mathbb{R}^{nN} as in Remark 3.3(i)) and Theorem 3.4 to obtain

$$\begin{aligned}
&\mathbb{P}\left(\phi\left(\bigcap_{i=1}^N B(X_i, r_i)\right) > s\right) \\
&= \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} H(x_1, \dots, x_N) \prod_{i=1}^N \prod_{j=1}^n h_{ij}(x_{ij}) dx_1 \dots dx_N \\
&\leq \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} H(x_1, \dots, x_N) \prod_{i=1}^N \prod_{j=1}^n h_{ij}^*(x_i) dx_1 \dots dx_N
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} H(x_1, \dots, x_N) \prod_{i=1}^N \mathbb{1}_{Q_n}(x_i) dx_1 \dots dx_N \\
&= \mathbb{P}\left(\phi\left(\bigcap_{i=1}^N B(Y_i, r_i)\right) > s\right).
\end{aligned}$$

□

Remark 3.9. One can adapt the latter argument to treat densities h_{ij} that are not necessarily bounded by the same value. In this case, h_{ij}^* is less peaked than $\|h_{ij}\|_\infty \mathbb{1}_{[-\frac{1}{2\|h_{ij}\|_\infty}, \frac{1}{2\|h_{ij}\|_\infty}]}$. Then the corresponding extremizers would be uniform measures on suitable coordinate boxes.

4 Wulff shapes and ball-polyhedra

Viewing a convex body as the intersection of its supporting halfspaces leads naturally to approximation by ball-polyhedra of large radii. More generally, one can work with Wulff shapes, which are defined as intersections of halfspaces. In this section, we make this connection explicit; detailed proofs are included for completeness and this will aid in interpreting the stochastic dominance discussed in the introduction. For background on Wulff shapes in Brunn-Minkowski theory and further references, see Schneider [36, §7.5] and for recent results see work of Böröczky, Lutwak, Yang and Zhang [8] and Schuster and Weberndorfer [37].

If $f : S^{n-1} \rightarrow \mathbb{R}$ is a positive continuous function, the Wulff shape $W(f)$ is defined by

$$W(f) = \bigcap_{\theta \in S^{n-1}} H^-(\theta, f(\theta)), \quad (4.1)$$

where

$$H^-(\theta, f(\theta)) = \{x \in \mathbb{R}^n : \langle x, \theta \rangle \leq f(\theta)\}. \quad (4.2)$$

Then $W(f)$ is a convex body with the origin in its interior. If K is a convex body with positive support function h_K , then $W(h_K) = K$.

With f as above and $R > \sup_{\theta \in S^{n-1}} f(\theta)$, we introduce a star body $A(f, R)$ by specifying its radial function:

$$\rho_{A(f, R)}(-\theta) = R - f(\theta) \quad (\theta \in S^{n-1}). \quad (4.3)$$

The role of $A(f, R)$ is described in the next result; as above, $\text{vr}(A(f, R))$ is the radius of a Euclidean ball with the same volume as $A(f, R)$. When f is the (positive) support function h_K of a convex body K we also write $A(K, R)$ instead of $A(h_K, R)$.

Proposition 4.1. Let $f : S^{n-1} \rightarrow \mathbb{R}$ be positive and continuous, $R > \sup_{\theta \in S^{n-1}} f(\theta)$ and $A(f, R)$ as in (4.3). Then, in the Hausdorff metric,

$$W(f) = \lim_{R \rightarrow \infty} \bigcap_{x \in A(f, R)} B(x, R), \quad (4.4)$$

and

$$R - \text{vr}(A(f, R)) \leq \int_{S^{n-1}} f(\theta) d\sigma(\theta); \quad (4.5)$$

moreover, equality holds as $R \rightarrow \infty$.

The proof of the proposition relies on the following lemmas.

Lemma 4.2. Let $N, n \geq 1$, $x_1, \dots, x_N \in \mathbb{R}^n$ and set $P = \text{conv}\{x_1, \dots, x_N\}$. Then for each $r > 0$,

$$\bigcap_{x \in P} B(x, r) = \bigcap_{i=1}^N B(x_i, r). \quad (4.6)$$

Proof of Lemma 4.2. Let $y \in \bigcap_{i=1}^N B(x_i, r)$ so that $|y - x_i| \leq r$ for each $i = 1, \dots, N$. Let $x \in P$ and write $x = \sum_{i=1}^N \alpha_i x_i$, where $\alpha_1, \dots, \alpha_N \geq 0$ and $\sum_{i=1}^N \alpha_i = 1$. Then

$$|y - x| = \left| \sum_{i=1}^N \alpha_i y - \sum_{i=1}^N \alpha_i x_i \right| \leq \sum_{i=1}^N \alpha_i |y - x_i| \leq r,$$

hence $y \in \bigcap_{x \in P} B(x, r)$. The reverse inclusion is trivial. \square

Lemma 4.3. Let $f : S^{n-1} \rightarrow \mathbb{R}$ be positive and continuous. Assume that $\theta_1, \dots, \theta_N$ are points on the sphere that do not lie on a hemisphere. Then

$$\bigcap_{i=1}^N H^-(\theta_i, f(\theta_i)) = \lim_{R \rightarrow \infty} \bigcap_{i=1}^N B(-(R - f(\theta_i))\theta_i, R), \quad (4.7)$$

where the convergence is in the Hausdorff metric.

Proof of Lemma 4.3. Fix $R > \sup_{\theta \in S^{n-1}} f(\theta)$. Set $L = \bigcap_{i=1}^N H^-(\theta_i, f(\theta_i))$. By definition of $A(f, R)$,

$$\bigcap_{i=1}^N B(-(R - f(\theta_i))\theta_i, R) \subseteq \bigcap_{i=1}^N H^-(\theta_i, f(\theta_i)). \quad (4.8)$$

Next note that for any $\theta \in S^{n-1}$,

$$\begin{aligned} B(-(R - f(\theta))\theta, R) &\supseteq \{x \in L : |x + (R - f(\theta))\theta|^2 \leq R^2\} \\ &= \left\{ x \in L : \langle x, \theta \rangle \leq \frac{2Rf(\theta) - f^2(\theta) - |x|^2}{2(R - f(\theta))} \right\} \end{aligned}$$

$$\begin{aligned}
&= \left\{ x \in L : \langle x, \theta \rangle \leq \frac{1 - \frac{f(\theta)}{2R} - \frac{|x|^2}{2Rf(\theta)}}{1 - \frac{f(\theta)}{R}} f(\theta) \right\} \\
&\supseteq \{ x \in L : \langle x, \theta \rangle \leq (1 - O(1/R))f(\theta) \},
\end{aligned}$$

where the implied constants in $O(1/R)$ depend only the extreme values of f on S^{n-1} . Combining this with (4.8), we get

$$(1 - O(1/R))L \subseteq \bigcap_{i=1}^N B(-(R - f(\theta_i))\theta_i, R) \subseteq L, \quad (4.9)$$

which gives the result. \square

Proof of Proposition 4.1. The map $\theta \mapsto -(R - f(\theta))\theta$ is a bijection between S^{n-1} and the boundary $\partial A(f, R)$ of $A(f, R)$. Therefore

$$\bigcap_{\theta \in S^{n-1}} B(-(R - f(\theta))\theta, R) = \bigcap_{x \in \partial A(f, R)} B(x, R) \quad (4.10)$$

$$= \bigcap_{x \in A(f, R)} B(x, R), \quad (4.11)$$

where the last equality is simply Lemma 4.2 applied on each line segment

$$P(\theta) = \text{conv}\{\rho_{A(f, R)}(\theta)\theta, -\rho_{A(f, R)}(-\theta)\theta\} \quad (\theta \in S^{n-1}).$$

Thus equality (4.4) follows from Lemma 4.3. Since $A(f, R)$ is a star body, we can use polar coordinates and Jensen's inequality to get

$$\begin{aligned}
\text{vr}(A(f, R)) &= \left(\int_{S^{n-1}} \rho_{A(f, R)}(-\theta)^n d\sigma(\theta) \right)^{1/n} \\
&= \left(\int_{S^{n-1}} (R - f(\theta))^n d\sigma(\theta) \right)^{1/n} \\
&\geq R - \int_{S^{n-1}} f(\theta) d\sigma(\theta).
\end{aligned}$$

Writing $\|f\|_1 = \int_{S^{n-1}} f(\theta) d\sigma(\theta)$, we can prove that asymptotic equality holds in the latter by Taylor expansion:

$$\begin{aligned}
\text{vr}(A(f, R)) &= R \left(\int_{S^{n-1}} \left(1 - \frac{nf(\theta)}{R} + O(1/R^2) \right) d\sigma(\theta) \right)^{1/n} \\
&= R \left(1 - \frac{n\|f\|_1}{R} + O(1/R^2) \right)^{1/n}
\end{aligned}$$

$$\begin{aligned}
&= R \exp \left(\frac{1}{n} \log \left(1 - \frac{n \|f\|_1}{R} + O(1/R^2) \right) \right) \\
&= R \exp \left(\frac{1}{n} \left(-\frac{n \|f\|_1}{R} + O(1/R^2) \right) \right) \\
&= R \exp \left(-\frac{\|f\|_1}{R} + O(1/R^2) \right) \\
&= R \left(1 - \frac{\|f\|_1}{R} + O(1/R^2) \right) \\
&= R - \|f\|_1 + O(1/R).
\end{aligned}$$

□

5 Randomized inequalities related to the generalized Urysohn inequality

In this section, we discuss and compare the randomized versions of the generalized Urysohn inequality. We start by sketching the proof of Böröczky and Schneider mentioned in the introduction. With μ_K and \mathcal{H}_K as above, we have

$$\mu_K(A) = \int_{S^{n-1}} \int_0^1 \mathbb{1}_{\{H(\theta, h_K(\theta) + t) \in A\}} dt d\sigma(\theta)$$

for Borel sets $A \subseteq \mathcal{H}_K$. For $\Theta = (\theta_1, \dots, \theta_N) \in (S^{n-1})^N$ and $t = (t_1, \dots, t_N) \in \mathbb{R}^N$, set

$$P(K, \Theta, t) = H^-(\theta_1, h_K(\theta_1) + t_1) \cap \dots \cap H^-(\theta_N, h_K(\theta_N) + t_N) \cap (K + B).$$

Write $d\Theta$ for $d\sigma(\theta_1) \dots d\sigma(\theta_N)$. For independent random hyperplanes H_1, \dots, H_N sampled according to μ_K , set

$$K^{(N)} = \bigcap_{i=1}^N H_i^- \cap (K + B).$$

Then

$$\mathbb{E} V_j \left(K^{(N)} \right)^{1/j} = \int_{(S^{n-1})^N} \int_{[0,1]^N} V_j(P(K, \Theta, t))^{1/j} dt d\Theta.$$

For convex bodies K and L in \mathbb{R}^n and $\alpha \in [0, 1]$, one has the following inclusion

$$(1 - \alpha)P(K, \Theta, t) + \alpha P(L, \Theta, t) \subseteq P((1 - \alpha)K + \alpha L, \Theta, t).$$

Thus

$$V_j(P((1-\alpha)K + \alpha L, \Theta, t))^{1/j} \geq (1-\alpha)V_j(P(K, \Theta, t))^{1/j} + \alpha V_j(P(L, \Theta, t))^{1/j}.$$

Then

$$\mathbb{E}V_j\left(\left[(1-\alpha)K + \alpha L\right]^{(N)}\right)^{1/j} \geq (1-\alpha)\mathbb{E}V_j\left(K^{(N)}\right)^{1/j} + \alpha\mathbb{E}V_j\left(L^{(N)}\right)^{1/j}.$$

Thus, $K \mapsto \mathbb{E}V_j(K^{(N)})^{1/j}$ is concave with respect to Minkowski addition and it is also rotation invariant and continuous with respect to δ^H . In particular, for any direction u , the Minkowski symmetral $M_u(K) = \frac{K + R_u(K)}{2}$ satisfies $\mathbb{E}V_j(M_u(K)^{(N)})^{1/j} \geq \mathbb{E}V_j(K^{(N)})^{1/j}$ (where, as above, R_u denotes reflection about u^\perp). A theorem of Hadwiger (e.g., [36]) implies there is a sequence of directions so that successive Minkowski symmetrizations about those directions converge to a Euclidean ball with the same mean width as K . This establishes (1.5).

Next, we prove the following extension of Corollary 1.2.

Corollary 5.1. *Let K be a convex body in \mathbb{R}^n with the origin in its interior, $R > 0$ and assume $K \subseteq B(0, R)$. Consider independent random vectors X_1, \dots, X_N sampled according to $\frac{1}{|A(K, R)|} \mathbb{1}_{A(K, R)}$ and Z_1, \dots, Z_N according to $\frac{1}{|rB|} \mathbb{1}_{rB}$, where $r = r(K, n, R)$ satisfies $|A(K, R)| = |rB|$. Let $\phi : \mathcal{K}^n \rightarrow (0, \infty)$ satisfy (a), (b) and (c) (as defined at the beginning of §3). Then for each $p \in \mathbb{R}$,*

$$\left(\mathbb{E}\phi\left(\bigcap_{i=1}^N B(X_i, R)\right)^p\right)^{1/p} \leq \left(\mathbb{E}\phi\left(\bigcap_{i=1}^N B(Z_i, R)\right)^p\right)^{1/p}. \quad (5.1)$$

Consequently, if ϕ is continuous with respect to δ^H , we get

$$\min_{x_1, \dots, x_N \in A(K, R)} \phi\left(\bigcap_{i=1}^N B(x_i, R)\right) \leq \min_{z_1, \dots, z_N \in rB} \phi\left(\bigcap_{i=1}^N B(z_i, R)\right). \quad (5.2)$$

Proof. Choose $\varepsilon > 0$ such that $B(0, \varepsilon) \subseteq K$ so that for any $x_1, \dots, x_N \in A(K, R)$, we have

$$B(0, \varepsilon) \subseteq \bigcap_{i=1}^N B(x_i, R) \subseteq B(x_1, R).$$

In particular, all the moments in (5.1) are positive and finite. The same argument applies to points in rB . The moment inequality (5.1) follows immediately from Theorem 3.1. If ϕ is continuous, then

$$\min_{x_1, \dots, x_N \in A(K, R)} \phi\left(\bigcap_{i=1}^N B(x_i, R)\right) = \lim_{p \rightarrow -\infty} \left(\mathbb{E}\phi\left(\bigcap_{i=1}^N B(X_i, R)\right)^p\right)^{1/p} \quad (5.3)$$

and the same holds for rB in place of $A(K, R)$ and Z_i for X_i , which gives (5.2). \square

For $N > n$ and $j \in \{1, \dots, n\}$, let

$$m_{j,N}(K) = \min \left\{ V_j \left(\bigcap_{i=1}^N H_i^- \right) : K \subseteq H_i^-, i = 1, \dots, N \right\}, \quad (5.4)$$

where H_i^- is the closed halfspace bounded by H_i that contains K . As a consequence of Corollary 5.1, we get the following, which is a special case of a result due to Schneider [35].

Corollary 5.2. *Let K be a convex body in \mathbb{R}^n , $N > n$ and $j \in \{1, \dots, n\}$. Then*

$$m_{j,N}(K) \leq m_{j,N}((w(K)/2)B). \quad (5.5)$$

Proof. Without loss of generality we will assume that the origin is an interior point of K . Choose $R > 0$ such that $K \subseteq B(0, R)$. Since $A(K, R)$ is star-shaped, for any $x \in A(K, R)$ the line through x and the origin intersects $A(K, R)$ in a line segment, the endpoints of which are on the boundary $\partial A(K, R)$. Hence quasi-concavity of V_j yields

$$\min_{x_1, \dots, x_N \in A(K, R)} V_j \left(\bigcap_{i=1}^N B(x_i, R) \right) = \min_{x_1, \dots, x_N \in \partial A(K, R)} V_j \left(\bigcap_{i=1}^N B(x_i, R) \right); \quad (5.6)$$

the same holds with rB in place of $A(K, R)$. Therefore, (5.2) implies

$$\min_{x_1, \dots, x_N \in \partial A(K, R)} V_j \left(\bigcap_{i=1}^N B(x_i, R) \right) \leq \min_{x_1, \dots, x_N \in rS^{n-1}} V_j \left(\bigcap_{i=1}^N B(x_i, R) \right), \quad (5.7)$$

hence

$$\min_{\theta_1, \dots, \theta_N \in S^{n-1}} V_j \left(\bigcap_{i=1}^N B(-(R - h_K(\theta_i))\theta_i, R) \right) \leq \min_{x_1, \dots, x_N \in rS^{n-1}} V_j \left(\bigcap_{i=1}^N B(x_i, R) \right).$$

By (4.9), we have

$$m_{j,N}(K) = \sup_{R>0} \min_{\theta_1, \dots, \theta_N \in S^{n-1}} V_j \left(\bigcap_{i=1}^N B(-(R - h_K(\theta_i))\theta_i, R) \right). \quad (5.8)$$

For $K = (w(K)/2)B$, we get

$$\begin{aligned} m_{j,N}((w(K)/2)B) &= \sup_{R>0} \min_{\theta_1, \dots, \theta_N \in S^{n-1}} V_j \left(\bigcap_{i=1}^N B(-(R - w(K)/2)\theta_i, R) \right) \\ &= \sup_{R>0} \min_{\theta_1, \dots, \theta_N \in S^{n-1}} V_j \left(\bigcap_{i=1}^N B(-r(K, n, R)\theta_i, R) \right), \end{aligned} \quad (5.9)$$

where the latter follows from the asymptotic equality in (4.5). The corollary now follows from (5.8) and (5.9). \square

When $N \rightarrow \infty$ in (5.5), we get (1.2). This indicates that the stochastic dominance in Theorem 1.1 for $f = \frac{1}{|A(K,R)|} \mathbb{1}_{A(K,R)}$ can be regarded as a distributional form of (1.2).

Remark 5.3. Schneider [35] proved a more general variant of (5.5) with V_j replaced by a function ϕ which is concave, monotone, upper semi-continuous and minimized over rotations of K . We do not pursue a more detailed comparison as this is not our main goal.

Schneider's result (5.5) is a companion to that of Macbeath for maximum volume simplices inscribed in convex bodies [27]. Using (5.5) for $j = n$ and the reverse Urysohn inequality due to Pisier [32] and Figiel and Tomczak-Jaegermann [16] we get the following.

Corollary 5.4. *Let K be a convex body in \mathbb{R}^n . Then there is a simplex S containing K such that*

$$V_n(S) \leq (C \log n)^n n^{\frac{n+1}{2}} V_n(K), \quad (5.10)$$

where C is an absolute constant.

The latter improves on a result of Kanazawa [23] who proved that $V_n(S) \leq n^{n-1} V_n(K)$, which extends a classical planar result of Gross [20] to higher dimensions.

Proof. As the problem is invariant under affine transformations, we may first apply the reverse Urysohn inequality due to Pisier and Figiel and Tomczak-Jaegermann (see [2, Theorem 6.5.4]) and assume that

$$\frac{w(K)}{w(B)} \leq C_1 \log n \left(\frac{V_n(K)}{V_n(B)} \right)^{1/n}, \quad (5.11)$$

where C_1 is an absolute constant. By Schneider's result (5.5) we have

$$m_{n,n+1}(K) \leq m_{n,n+1}(B) \left(\frac{w(K)}{w(B)} \right)^n. \quad (5.12)$$

On the other hand,

$$m_{n,n+1}(B) = \frac{n^{\frac{n}{2}} (n+1)^{\frac{n+1}{2}}}{n!}. \quad (5.13)$$

The result now follows from (5.12) and (5.13). \square

5.1 Further connections to Minkowski symmetrization

In this section, we discuss the effect of Minkowski symmetrization of K on $A(K, R)$. We show that one can obtain (5.2) via Minkowski symmetrization as well. If K and L are convex bodies, the equality $h_{(K+L)/2} = (h_K + h_L)/2$ implies

$$\rho_{A(\frac{K+L}{2}, R)} = \frac{1}{2}(\rho_{A(K, R)} + \rho_{A(L, R)}).$$

In other words, the map $K \mapsto A(K, R)$ takes Minkowski sums to radials sums. In particular, if $u \in S^{n-1}$ and $M_u(K)$ is the Minkowski symmetrization of K about u^\perp , then $A(M_u(K), R)$ is the star-body with radial function $\frac{1}{2}(\rho_{A(K, R)} + \rho_{A(R_u(K), R)})$.

Assume now that $\theta_1, \dots, \theta_N \in S^{n-1}$. Then

$$\begin{aligned} & \bigcap_{i=1}^N B(-(R - h_{M_u(K)}(\theta_i))\theta_i, R) \\ &= \bigcap_{i=1}^N B(\rho_{A(M_u(K), R)}(\theta_i)\theta_i, R) \\ &\supseteq \frac{1}{2} \bigcap_{i=1}^N B(\rho_{A(K, R)}(\theta_i)\theta_i, R) + \frac{1}{2} \bigcap_{i=1}^N B(\rho_{A(R_u(K), R)}(\theta_i)\theta_i, R) \\ &= \frac{1}{2} \bigcap_{i=1}^N B(-(R - h_K(\theta_i))\theta_i, R) + \frac{1}{2} \bigcap_{i=1}^N B(-(R - h_{R_u(K)}(\theta_i))\theta_i, R) \\ &= \frac{1}{2} \bigcap_{i=1}^N B(-(R - h_K(\theta_i))\theta_i, R) + \frac{1}{2} R_u \left(\bigcap_{i=1}^N B(-(R - h_K(R_u^t \theta_i))R_u^t \theta_i, R) \right), \end{aligned}$$

where R_u^t is the transpose of R_u . We now use quasi-concavity of ϕ and rotation invariance to get

$$\phi \left(\bigcap_{i=1}^N B(-(R - h_{M_u(K)}(\theta_i))\theta_i, R) \right) \geq \min_{\theta_1, \dots, \theta_N \in S^{n-1}} \phi \left(\bigcap_{i=1}^N B(-(R - h_K(\theta_i))\theta_i, R) \right). \quad (5.14)$$

As mentioned above, given a convex body K , a theorem of Hadwiger implies that there is a sequence of directions so that successive Minkowski symmetrizations about those directions converge to a Euclidean ball with the same mean width as K . Combining this with inequality (5.14), we get another proof of (5.2).

5.2 Connection between random ball-polyhedra and random convex hulls

As mentioned already, the inequality for random ball-polyhedra obtained by taking $j = n$ in (1.6) implies Urysohn's inequality, and so does the inequality for random convex hulls when $j = 1$ in (1.4). Here we show that the former

implies the latter. The proof uses the following theorem of Gorbovickis [18, Theorem 4].

Theorem 5.5. *Let $x_1, \dots, x_N \in \mathbb{R}^n$ where $n \geq 2$. Then the following asymptotic equality holds as $R \rightarrow \infty$:*

$$\left| \left(\bigcap_{i=1}^N B(x_i, R) \right) \right| = \omega_n R^n - n\omega_n w(\text{conv}\{x_1, \dots, x_N\}) R^{n-1} + o(R^{n-1}). \quad (5.15)$$

Assume that K is a convex body in \mathbb{R}^n with $|K| = |B|$. Sample independent random vectors X_1, \dots, X_N in K and Z_1, \dots, Z_N in B according to their respective uniform probability measures. For each fixed value of X_1, \dots, X_N , Theorem 5.5 implies

$$n\omega_n w(\text{conv}\{X_1, \dots, X_N\}) = R - R^{-(n-1)} \left| \left(\bigcap_{i=1}^N B(X_i, R) \right) \right| + o(1), \quad (5.16)$$

as $R \rightarrow \infty$. By compactness of K , we can use dominated convergence to conclude

$$n\omega_n \mathbb{E} w(\text{conv}\{X_1, \dots, X_N\}) = R - R^{-(n-1)} \mathbb{E} \left| \left(\bigcap_{i=1}^N B(X_i, R) \right) \right| + \mathbb{E} o(1),$$

as $R \rightarrow \infty$. By continuity of the volume of the intersection and the mean width, the quantity $\mathbb{E} o(1)$ is also of the form $o(1)$. The same argument applies to Z_1, \dots, Z_N . By Theorem 3.1, we get

$$\mathbb{E} w(\text{conv}\{X_1, \dots, X_N\}) \geq \mathbb{E} w(\text{conv}\{Z_1, \dots, Z_N\}),$$

which is equivalent to the $j = 1$ case in (1.4).

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